International Journal of Theoretical Physics, Vol. 7, No. 4 (1973), pp. 281–285

Comments on the Clock Paradox

MENDEL SACHS

Department of Physics, State University of New York, Buffalo, N.Y.

Received: 26 *May* 1972

I should like to briefly comment on a recent paper published in this journal by my colleagues T.-Y. Wu & Y. C. Lee (1972) on the clock paradox. They claim to carry out an exact calculation of the comparison of total proper times of two systems that are initially in the same inertial frame, one then making a set of changes from this frame to take a round-trip journey, eventually returning to the frame of the other. Their conclusion, which agrees with the claim of the majority, is that a twin who takes the trip will return having aged less than his brother, and that both twins will agree on the precise quantity of age retardation of the traveller, relative to the 'stay-at-home' brother.

I do not believe that these authors have in fact carried out an exact, nor an unambiguous treatment of this problem, nor that their conclusions are justified, even in an approximate sense. Their method of analysis is perhaps applicable to some problems in special relativity theory, but not so to the clock problem, which I contend must necessarily entail an incorporation of nonuniform motion in the proper way, according to general relativity theory as a crucial ingredient.

First, recall the source of the logical paradox in this problem. It is this: If the time parameter in any frame directly correlates with the physical process of aging in that frame, *as viewed by any particular observer*, then the time contraction from one frame to another relatively moving one implies that the physical system is aging more slowly in the moving frame. The logical paradox comes up because 'moving' is purely subjective in relativity theory—that is to say, an observer in the frame that was previously called 'moving' can equally be called 'stationary', observing the observer who was previously 'stationary' to be 'moving', without altering any of the physical description, or any of its objective conclusions. Thus, both observers would claim that the other is aging more slowly! Now if both observers' statements are scientifically valid, then when one of the twins returns home, his brother would conclude that he is both older and younger than he is! This is a logical paradox—i.e. it is nonsense—and therefore must be removed.

To resolve the paradox, the theory must predict instead that both brothers agree on a conclusion (qualitatively and quantitatively) about the outcome of the experience. There are two possible choices: (1) either one of the twins will be younger after the journey is over (both agreeing on the precise amount of age discrepancy) or (2) there is no asymmetric aging.

Einstein originally argued for choice (1)—as do most contemporary physicists (and as Wu and Lee do in their paper). In a recent paper (Sachs, 1971) I have argued that choice (2) is the only one that is implied when the theory of relativity is fully exploited—both from the logical point of view and from the view of an exact, unambiguous mathematical treatment of this problem.

Einstein argued that the paradox would be resolved when the actual asymmetry is taken into account, regarding the motion of a space traveller away from the earth. That is, the man in the rocket ship would see the earth, as well as his brother, moving away from him as his rocket engines blast off. The 'stationary' man at the launching site, on the other hand, would see his brother moving away from him and a stationary earth. In one case, then, the earth is in motion, while in the other, it is not—thereby describing an asymmetry in the respective motions of the twins relative to their surroundings. How should one handle this comparison of descriptions unambiguously? Secondly, why should a resolution of the paradox not exist in terms of a universe of two equally massive systems moving relative to each other—i.e. where the earth would not be involved?

Einstein suggested that to introduce the asymmetry taking into account the motion of the rocket in the field of the earth, one may use the principle of equivalence to derive the time of turn-around of the rocket ship. This, in turn, adds a compensating term in the earth observer's measurements that is not added in the rocket ship observer's measurements, and predicts that both twins will then agree that it is the rocket ship traveller who will age less than his brother during the round trip journey. The equation of motion that Einstein suggests for the turn-around period is, according to the principle of equivalence, $g =$ constant, g being the acceleration due to gravity (see Tolman, 1934).

I argued in my paper (Sachs, 1971) that while this procedure does give a compensating term so that both observers would agree that only one of them should be younger after the trip is over, the prediction is quantitatively ambiguous because the equation of motion, $g =$ const., is only the Newtonian limit of more general equations of motion of general relativity theory (the geodesic equation entails ten potential functions of space-time coordinates). When the approximation used for this equation of motion is relaxed, there is no reason why in the more precise description the prediction would remain that the two twins would be in agreement on the quantitative value of asymmetric aging, if it indeed happens. That is, there is no proof that the coefficients of v/c to higher powers than 2, would agree numerically in comparing the predictions of each of the twins.

The Wu–Lee result is more detailed than this, but it is also based on an approximation that yields an ambiguous conclusion. They consider the *Lorentz covariant* description of special relativity, where the proper time is $\tau = t(1-(v/c)^2)^{1/2}$, where v is the constant velocity of the relative motion. They then extend this transformation by keeping its form but letting v become variable. They then integrate $d\tau = dt(1-(v(x,t)/c)^2)^{1/2}$ over the whole closed path to get the proper time elapsed. But this is not a valid transformation for general relativity—i.e. it does not leave invariant the metric $ds^2 = g^{\gamma\delta}(x,t)\,dx_\gamma\,dx_\delta$. They are, in fact, led back to a space-time that is equivalent to the Euclidean metric of special relativity, $dx_0{}^2 - dr^2$.

Thus, Wu and Lee are actually talking about the motions in the tangent planes (with Euclidean geometry) at the points in space-time where the nonuniform motion really requires a *curved* (non-Euclidean) space-time. At each point, then, they are re-orienting the tangent planes to simulate a change in the orientation that is evolving in the actual curved space. But this procedure is ambiguous because one does not know here precisely how much the orientation of these planes must change, as one proceeds continuously along a path. The actual connection between the orientations of these tangent planes at continuously connected space-time points is, in fact, specified by the affine-connection of the Riemannian space-time. But Wu and Lee do not incorporate the affine connection into their analysis! The actual problem requires some non-uniform motion relative to a fixed observer, which in turn means that the space-time is *curved.* If it is curved somewhere, then the space-time is curved everywhere, and this curvature is *essential to an unambiguous analysis* that would resolve the clock paradox!

According to the final remark in their paper, Wu and Lee agree with this fact, that they are not really analysing the problem in terms of a curved space-time, as would be rigorously required if one were incorporating the non-uniform motion in an exact way. But, in contrast with their comment, this does *not* mean that the clock paradox has 'no clear and exact meaning in general relativity'. Indeed, the clock paradox has a well-defined meaning in general relativity, and as a logical paradox it must be removed in order to make sense of the theory! As Einstein himself argued, there would be no resolution of the clock *paradox* without going to the general relativity description!

The calculational procedure that Wu and Lee follow may be useful if one were considering effects that are not sensitive to the differential changes in the metric space—such as the local special relativistic predictions of the conserved energy, momentum, etc. But they are investigating an effect that *is due to* making changes from one inertial frame to another—an effect (if it is present) that must be sensitive to the differential geometry. It is then fallacious to conclude results of an analysis based on a flat space geometry as representing physical effects that are due to the *differential changes* in the geometry of a Riemann space. I contend, then, that the Wu–Lee result is not only not exact (as they claim it is), it is not even approximately correct. There is no indication in their analysis that they have 'resolved the clock paradox'.

I have the following additional comments on their paper: 1. It is not true, as Wu and Lee assert, that the symmetry group that leaves invariant $ds^2 = g^{\gamma\delta} dx_\gamma dx_\delta$ (the 'Einstein group') includes the Lorentz group in an exact sense. The latter refers to the symmetry of special relativity—corresponding to a flat space geometry. But if the space is curved, then it is curved everywhere—even though, for calculational purposes, one can approximate the curved space in the local domain with the space in a tangent plane at that point. Thus, there is no real incorporation of the Lorentz group in the Einstein group—it is only valid in an approximate sense. I have discussed this point in more detail in Sachs (1969). In this case, then, it is logically and mathematically fallacious to claim the validity of a result derived with a flat space geometry that is implicitly *due to* the curvature of space-time! 2. The total elapsed proper time between P_1 and P_2 in space-time is defined, in an *exact* sense, by the geodesic $\int_{P_1}^{P_2} ds$. This path, in turn, follows from the solutions of the geodesic equation. The exact solutions of the Einstein field equations, $g^{\gamma\delta}$, then correspond to a given geodesic. But for any other energy-momentum tensor source $T^{\gamma\delta}$ of Einstein's equations, there is a different set of solutions, $g^{\gamma\delta}$, and consequently a different set of geodesic paths. Thus to treat the clock problem, in particular, *exactly*, one must consider two distinct geodesic paths, connected at the end points. The difference between these two geodesics is due to the difference in the source terms $T^{\gamma\delta}$—one incorporating the energy-momentum associated with the rocket engines, e.g., that propelled one twin away from earth and then brought him back again, and the other not involving this extra contribution. Both paths are then treated separately as closed systems—there is no 'external' force involved in the problem.

Different source terms, $T_1^{\lambda\delta}$, $T_2^{\lambda\delta}$, then give different solutions $g_1^{\lambda\delta}$, $g_2^{\lambda\delta}$ in space-time, that correspond to the different geodesic paths that are to be connected at the end points. There is no reason why the two geodesic paths, expressed in terms of the distinct metric tensor solutions, cannot cross at the two end points. Thus, to treat one path as a geodesic and the other not, as Wu and Lee assert, is to use an approximation to the exact problem that, in fact, would not always yield an unambiguous result. In my analysis (Sachs, 1971) I compared two *exact* geodesics, connected at the end points in space-time. I found from a functional analysis of the most general representation of general relativity theory (in terms of a quaternion field representation of the space-time metric) that the total proper times between arbitrary points in space-time (away from possible singularities†) is path-independent.

† Because of the role of continuity in relativity theory, and the definition of the metric tensor in terms of the differential geometry of space-time, I would contend that $g^{\gamma\delta}$ must necessarily be nonsingular functions of the space-time coordinates. Singularities can, of course, be utilized as convenient mathematical representations of the behavior of nonsingular fields in particular localities—e.g. representing the actually continuous energy-momentum tensor density of the sun by a delta function source, or representing the boundary conditions on the metric field, that relates to the distribution of distant matter, in terms of singularities at infinity. But these are only mathematical devices to replace a continuous source field. The requirement of an actual continuity in the representation

It then followed, with the correlation of proper time and the physical aging in the proper frame, that the path-independence of the total proper times for two *physically identical* 'clocks' implies the path-independence of their aging over the respective paths in space-time. Thus it was predicted from an exact, unambiguous analysis that asymmetric aging is not a predicted consequence of the theory of relativity.

References

Einstein, A. and Rosen, N. (1935). *Physical Review*, 48, 73.
Sachs, M. (1969). *Physics Today*, 22, 51.
Sachs, M. (1971). *Physics Today*, 24, 23.
Tolman, R. T. (1934). *Relativity, Thermodynamics and Cosmology*. Oxford University Press.
Wu, T.-Y. and Lee, Y. C. (1972). *International Journal of Theoretical Physics*, Vol. 5, No. 5, p. 307.

of matter was recognized at an early stage of development of the theory of general relativity (see, e.g. Einstein & Rosen, 1935).

In any case, the clock problem can be formulated in general relativity theory whether or not one should take singularities in the metric field to be real. Should such singularities be accepted as real (rather than merely mathematical devices to represent continuous matter) then my analysis would predict an asymmetric aging if the rocket traveller should travel around a real singular source and no asymmetric aging if the trip is into a region of space where the traveller did not go around a singular source!